Supplementary material for the paper: Root$-n$ consistent estimation of the marginal density in semiparametric autoregressive time series models

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6 Proof of assertions (13) and (14) in the proof of Theorem 1

6.1 Proof of assertion (13)

We set for $j \in \mathbb{N}$,

$$\mathcal{H}_{nj} = \sigma\left(\varepsilon_s, \varepsilon^{(s)} : s \geq n' - j\right).$$

For this part we set

$$Z_i(s,v) = \frac{1}{\sigma_{i\ell}} K_b\left[\frac{v - m_{i\ell}}{\sigma_{i\ell}} - s\right],$$

$\overline{Z}_i(s,v) = Z_i(s,v) - \mathbb{E}\left[Z_i(s,v)\right]$ and

$$\begin{aligned} S_{v,ij} &= A_{v,i(n'-j)} - \mathbb{E}\left(A_{v,i(n'-j)} \middle| \varepsilon_{n'-j}\right) \\ &- \mathbb{E}\left[A_{v,i(n'-j)} - \mathbb{E}\left(A_{v,i(n'-j)} \middle| \varepsilon_{n'-j}\right) \middle| \mathcal{H}_{n(j-1)}\right]. \end{aligned}$$

Using the independence properties, observe that

$$\begin{aligned} S_{v,ij} &= A_{v,i(n'-j)} - \frac{1}{\sigma_{i\ell}} \int K(w) f_\varepsilon\left[\frac{v - m_{i\ell}}{\sigma_{i\ell}} - bw\right] dw \\ &- \int K(h) g_v\left(\varepsilon_{n'-j} + bh\right) dh + \mathbb{E}\left[A_{v,i(n'-j)}\right]. \end{aligned}$$

From the first writing of $S_{v,ij}$, we have for $v, \bar{v} \in I$,

$$\begin{aligned} &\mathbb{E}\left|\sum_{j=\ell}^{n'-1} \sum_{i=n'-j-\ell}^{n'} \left[S_{v,ij} - S_{\bar{v},ij}\right]\right|^2 \\ \leq\ & 2\sum_{j=\ell}^{n'-1} \int \mathbb{E}\left|\sum_{i=n'-j-\ell}^{n'} \left[Z_i(s,v) - Z_i(s,\bar{v})\right]\right|^2 f_\varepsilon(s) ds. \end{aligned}$$

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Moreover, for $s \in \mathbb{R}$, we have, using $\ell-$dependence,

$$\mathbb{E}\left|\sum_{i=n'-j-\ell}^{n'}\left[Z_i(s,v)-Z_i(s,\bar{v})\right]\right|^2 \le \ell\sum_{h=1}^{\ell}\sum_{g=0}^{k-1}\mathbb{E}\left|Z_{h+g\ell}(s,v)-Z_{h+g\ell}(s,\bar{v})\right|^2.$$

Moreover, we have

$$\begin{aligned}
&\mathbb{E}\int\left|Z_i(s,v)-Z_i(s,\bar{v})\right|^2 f_\varepsilon(s)ds\\
\le\ & 2\mathbb{E}\int\left|\frac{1}{\sigma_{i\ell}}K_b\left[\frac{v-m_{i\ell}}{\sigma_{i\ell}}-s\right]-\frac{1}{\sigma_{i\ell}}K_b\left[\frac{\bar{v}-m_{i\ell}}{\sigma_{i\ell}}-s\right]\right|^2 f_\varepsilon(s)ds\\
\le\ & \frac{C|v-\bar{v}|^{\delta_1}}{b^{1+\delta_1}}\int\left|K(z)-K\left(\frac{\bar{v}-v}{b\sigma_{i\ell}}+z\right)\right|^{2-\delta_1} f_\varepsilon\left(\frac{v-m_{i\ell}}{\sigma_{i\ell}}-bz\right)dz\\
\le\ & \frac{C|v-\bar{v}|^{\delta_1}}{b^{1+\delta_1}}\int\int \mathbb{1}_{w\in[z,z+\frac{\bar{v}-v}{b\sigma_{i\ell}}]}|K'(w)|^{2-\delta_1}dwdz\\
\le\ & \frac{C}{b^{2+\delta_1}}|v-\bar{v}|^{1+\delta_1}.
\end{aligned}$$

Then we get,

$$\frac{1}{n^3}\mathbb{E}\left|\sum_{j=\ell}^{n'-1}\sum_{i=n'-j-\ell}^{n'}\left[S_{v,ij}-S_{\bar{v},ij}\right]\right|^2 \le \frac{C\ell}{nb^{2+\delta_1}}|v-\bar{v}|^{1+\delta_1}. \tag{20}$$

Note that $\frac{\ell}{nb^{2+\delta_1}} \to 0$. Moreover, it is easily seen that

$$\frac{1}{n^3}\mathbb{E}\left|\sum_{j=\ell}^{n-1}\sum_{i=n-j-\ell}^{n}S_{v,ij}\right|^2 \le \frac{C\ell}{nb^2} = o(1).$$

This means that $G_n : v \mapsto \frac{1}{n^{3/2}}\sum_{j=\ell}^{n-1}\sum_{i=n-j-\ell}^{n}S_{v,ij}$ converges pointwise to 0 in probability. Note that G_n is a random function taking values in $\mathcal{C}(I)$, the space of real-valued and continuous functions defined on the compact interval I. From (20) and the Kolmogorov-Chentsov tightness criterion in $\mathcal{C}(I)$ (see for instance Kallenberg (1997), Corollary 14.9), we (13).

6.2 Proof of assertion (14)

We have to prove the following convergence in probability:

$$\frac{1}{n^{3/2}}\sup_{v\in I}\left|\sum_{(i,j)\in\mathcal{I}_n}\int K(h)\left[g_v(\varepsilon_j+bh)-g_v(\varepsilon_j)\right]dh\right| = o_{\mathbb{P}}(1). \tag{21}$$

The proof is divided into two parts. In the first part, we show that

$$\frac{1}{n^{3/2}}\sup_{v\in I}\left|\sum_{(i,j)\in\mathcal{I}_n}\int K(h)\mathbb{E}\left[g_v(\varepsilon_j+bh)-g_v(\varepsilon_j)\right]dh\right| = o_{\mathbb{P}}(1). \tag{22}$$

We have

$$\begin{aligned}\int K(h)\mathbb{E}\left[g_v(\varepsilon_j+bh)-g_v(\varepsilon_j)\right]dh &= \int\int K(h)\left[g_v(u+bh)-g_v(u)\right]f_\varepsilon(u)dudh\\ &= \int\int K(h)g_v(z)\left[f_\varepsilon(z-bh)-f_\varepsilon(z)\right]dhdz.\end{aligned}$$

Using assertion (11) given in the paper, the condition $\sqrt{n}b^2\to 0$ and the fact that $\sup_{v\in I}\int g_v(z)dz\leq C\sup_{v\in I}\int f_{\frac{v-m_i}{\sigma_i}}(z)dz\leq C$, we get (22). To show (21), it remains to show that

$$\begin{aligned}&\frac{1}{n^{3/2}}\sup_{v\in I}\left|\sum_{(i,j)\in\mathcal{I}_n}\int K(h)\left[\overline{g_v(\varepsilon_j+bh)}-\overline{g_v(\varepsilon_j)}\right]dh\right|\\ &=\frac{1}{\sqrt{n}}\sup_{v\in I}\left|\sum_{j=1}^{n'}c_{n,j}\int K(h)\left[\overline{g_v(\varepsilon_j+bh)}-\overline{g_v(\varepsilon_j)}\right]dh\right| \qquad (23)\\ &=o_{\mathbb{P}}(1),\end{aligned}$$

where $c_{n,j}=\frac{1}{n}\sum_{i=1}^{n'}\mathbb{1}_{|i-j|\geq\ell}$.

1. To show (23) when I is the singleton $\{v\}$, we use Jensen inequality and the fact that the translations are continuous in $\mathbb{L}^2(f_\varepsilon)$ for any function in $\mathbb{L}^2(\mu)$. More precisely, we have

$$\begin{aligned}&n^{-3}\mathbb{E}\left|\sum_{(i,j)\in\mathcal{I}_n}\int K(h)\left[\overline{g_v(\varepsilon_j+bh)}-\overline{g_v(\varepsilon_j)}\right]dh\right|^2\\ \leq\;& n^{-1}\sum_{j=1}^{n'}c_{n,j}^2\int\left|\int K(h)\left[g_v(u+bh)-g_v(u)\right]dh\right|^2 f_\varepsilon(u)du\\ \leq\;& n'/n\int\int K(h)\left[g_v(u+bh)-g_v(u)\right]^2 f_\varepsilon(u)dudh\\ \leq\;& \sup_{|z|\leq b}\int\left[g_v(u+z)-g_v(u)\right]^2 f_\varepsilon(u)du\\ =\;& o_{\mathbb{P}}(1).\end{aligned}$$

2. To show (23) when I is a compact interval not reduced to one point, we proceed in two parts.

   (a) We first show that

$$\frac{1}{\sqrt{n}}\sup_{v\in I}\left|\sum_{j=1}^{n'}\int K(h)\left[g_v\left(\varepsilon_j+bh\right)-g_v(\varepsilon_j)\right]dh\right|=o_{\mathbb{P}}(1). \qquad (24)$$

   To this end, we will use Lemma 19.34 in van der Vaart (1998). We set

$$g_{n,v}(x)=\int K(h)\left[g_v(x+bh)-g_v(x)\right]dh.$$

For the family $\mathcal{G}_{n,I} = \{g_{n,v} : v \in I\}$ of functions, we consider the envelope function G_n defined by

$$G_n(x) = \int K(h) \left[G(x + bh) + G(x) \right] dh,$$

where G is defined in assumption **A7**. For bounding the bracketing numbers of this family, we first observe that if $[f_1, f_2]$ is an $\epsilon-$bracketing in $\mathcal{G}_I$, then $f_1 \leq g_v \leq f_2$ (i.e $\int (f_2 - f_1)^2 d\mu < \epsilon^2$) entails that

$$\begin{aligned} f_{n,1}(x) &= \int K(h) \left[f_1(x + bh) - f_2(x) \right] dh \\ &\leq g_{n,v}(x) \leq \int K(h) \left[f_2(x + bh) - f_1(x) \right] dh = f_{n,2}(x). \end{aligned}$$

Moreover, one can show that

$$\int \left| f_{n,2}(x) - f_{n,1}(x) \right|^2 f_\varepsilon(x) dx \leq 4\epsilon^2.$$

Then, we have

$$N_{[]} \left(\sqrt{2}\epsilon, \mathcal{G}_{n,I}, \mathbb{L}^2(\varepsilon_0) \right) \leq N_{[]} \left(\epsilon, \mathcal{G}_I, \mathbb{L}^2(\mu) \right)$$

and the bracketing numbers of the family $\mathcal{G}_{n,I}$ are of polynomial decay. Next we show that

$$\sup_{v \in I} \left| \int g_{n,v}(x)^2 f_\varepsilon(x) dx \right| = o_{\mathbb{P}}(1). \tag{25}$$

To show (25), we consider for a given $\epsilon > 0$, some brackets $I_1, I_2, \ldots, I_T$ that cover $\mathcal{G}_I$. For each integer $1 \leq p \leq T$, we consider an element $g_{v_p} \in I_p$. Then, for $1 \leq p \leq T$, if $I_p = \left[f_1^{(p)}, f_2^{(p)} \right]$, we set $I_{n,p} = \left[f_{n,1}^{(p)}, f_{n,2}^{(p)} \right]$, with

$$f_{n,1}^{(p)}(x) = \int K(h) \left[f_1^{(p)}(x + bh) - f_2^{(p)}(x) \right] dh,$$

$$f_{n,2}^{(p)}(x) = \int K(h) \left[f_2^{(p)}(x + bh) - f_1^{(p)}(x) \right] dh.$$

Then, if $g_v \in I_p$, we have $g_{n,v} \in I_{n,p}$ and

$$\begin{aligned} \int g_{n,v}^2(x) f_\varepsilon(x) dx &\leq 2 \int \left[g_{n,v}(x) - g_{n,v_p}(x) \right]^2 f_\varepsilon(x) dx + 2 \int g_{n,v_p}^2(x) f_\varepsilon(x) dx \\ &\leq 4\epsilon^2 + 2 \max_{1 \leq p \leq T} \int g_{n,v_p}^2(x) f_\varepsilon(x) dx. \end{aligned}$$

Since for each $v \in I$, we have $\int g_{n,v}(x)^2 f_\varepsilon(x) dx = o(1)$, we conclude that

$$\limsup_{n \to \infty} \int g_{n,v}^2(x) f_\varepsilon(x) dx \leq 4\epsilon^2.$$

Since ϵ is arbitrary, we conclude (25).

Moreover, setting, for $\kappa > 0$, $a_n(\kappa) = \kappa / Log\left[N_{[]}\left(\kappa, \mathcal{G}_{n,I}, \mathbb{L}^2(\varepsilon_0)\right)\right]$, we have

$$\begin{aligned}\sqrt{n}\mathbb{E}G_n(\varepsilon_0)\mathbb{1}_{G_n(\varepsilon_0)>\sqrt{n}a_n(\kappa)} &\leq \frac{\sqrt{n}}{\left(\sqrt{n}a_n(\kappa)\right)^{1+o}}\mathbb{E}G_n(\varepsilon_0)^{2+o} \\ &\leq \frac{2}{n^{o/2}a_n(\kappa)^{1+o}}\int G(x)^{2+o}d\mu(x).\end{aligned}$$

Here $Log(x) = \log(x) \wedge 1$. Then, from (25), one can choose $\kappa^2 = \kappa_n^2 \geq \int g_{n,v}(x)^2 f_\varepsilon(x)dx$ such that $\kappa \to 0$ and $n^{o/2}a_n(\kappa)^{1+o} \to \infty$. Hence, from Lemma 19.34 in van der Vaart (1998), we deduce that

$$\frac{1}{\sqrt{n}}\mathbb{E}\sup_{v\in I}\left|\sum_{j=1}^{n'}\overline{g_{n,v}(\varepsilon_j)}\right| = o_P(1)$$

and hence (24).

(b) Finally, we have

$$\begin{aligned}&\sup_{v\in I}\left|n^{-3/2}\sum_{(i,j)\in\mathcal{I}_n} g_{n,v}(\varepsilon_j) - \frac{1}{\sqrt{n}}\sum_{j=1}^{n'} g_{n,v}(\varepsilon_j)\right| \\ \leq\ & \frac{1}{\sqrt{n}}\sum_{j=1}^{n'}|c_{n,j}-1|\int K(h)\left[G(\varepsilon_j+bh)+G(\varepsilon_j)\right]dh \\ \leq\ & \frac{1}{n}\sum_{j=1}^{n'}\int K(h)\left|G(\varepsilon_j+bh)+G(\varepsilon_j)\right|dh\cdot\frac{n-n'+2\ell-1}{\sqrt{n}}.\end{aligned}$$

Using the fact that $\ell \sim n^t$ with $t < 1/2$, we deduce from assumption **A7** that

$$\sup_{v\in I}\left|n^{-3/2}\sum_{(i,j)\in\mathcal{I}_n} g_{n,v}(\varepsilon_j) - \frac{1}{\sqrt{n}}\sum_{j=1}^{n'} g_{n,v}(\varepsilon_j)\right| = o_{\mathbb{P}}(1).$$

From the last convergence and from (24), we deduce (23).

7 Proof of Corollary 1

From Theorem 1, Theorem 2 and assumption **A8**, we have

$$\sqrt{n}\left[\hat{f}_X(v) - f_X(v)\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\overline{M}_{i,v} + o_{\mathbb{P}}(1).$$

The first part of the corollary concerns the convergence of the finite dimensional distributions. Note also that the previous convergence is uniform if assumption **A7** holds true. Convergence of finite dimensional distributions is straightforward using a central limit theorem for weakly dependent time series. For instance, the central limit theorem given in Zhao (2010), Theorem 3, applies in our case. For the uniform convergence, it remains to show the tightness of the empirical process

$G_n : v \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^n \overline{M}_{i,v}$. Since we have assumed **A7**, it is only necessary to study the tightness in $\mathcal{C}(I)$, the space of real-valued and continuous function defined on I, of

$$\widetilde{G}_n : v \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\frac{1}{\sigma_i} f_\varepsilon \left(\frac{v - m_i}{\sigma_i} \right) - \mathbb{E} \frac{1}{\sigma_i} f_\varepsilon \left(\frac{v - m_i}{\sigma_i} \right) \right].$$

To this end, we use the Kolmogorov-Chentsov criterion (see for instance Kallenberg (1997), Corollary 14.9). If $v, \bar{v} \in I$ satisfy $v \leq \bar{v}$, we have from Jensen inequality,

$$\begin{aligned} \mathbb{E} \left| \widetilde{G}_n(v) - \widetilde{G}_n(\bar{v}) \right|^2 &\leq |v - \bar{v}| \int \mathbb{E} \left| \widetilde{G}'_n(u) \right|^2 du \\ &\leq |v - \bar{v}|^2 \sup_{u \in I} \mathbb{E} \left| \widetilde{G}'_n(u) \right|^2. \end{aligned}$$

Then the tightness will follow if we show that

$$\sup_{u \in I} \mathbb{E} \left| \widetilde{G}'_n(u) \right|^2 = O(1). \tag{26}$$

Note that

$$\widetilde{G}'_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\frac{1}{\sigma_i^2} f'_\varepsilon \left(\frac{u - m_i}{\sigma_i} \right) - \mathbb{E} \frac{1}{\sigma_i^2} f'_\varepsilon \left(\frac{u - m_i}{\sigma_i} \right) \right].$$

Moreover, for $i \leq j$, $v \in I$ and $\ell = j - i$, we have

$$\begin{aligned} &\operatorname{Cov} \left[\frac{1}{\sigma_i^2} f'_\varepsilon \left(\frac{v - m_i}{\sigma_i} \right), \frac{1}{\sigma_j^2} f'_\varepsilon \left(\frac{v - m_j}{\sigma_j} \right) \right] \\ \leq\ & \frac{\|f'_\varepsilon\|_\infty}{\gamma^2} \mathbb{E} \left| \frac{1}{\sigma_j^2} f'_\varepsilon \left(\frac{v - m_j}{\sigma_j} \right) - \frac{1}{\sigma_{j\ell}^2} f'_\varepsilon \left(\frac{v - m_{j\ell}}{\sigma_{j\ell}} \right) \right|. \end{aligned}$$

Then using assumption **A2** and assumption **A5**, we deduce that there exist $C > 0$ such that for all $i \leq j$,

$$\operatorname{Cov} \left[\frac{1}{\sigma_i^2} f'_\varepsilon \left(\frac{v - m_i}{\sigma_i} \right), \frac{1}{\sigma_j^2} f'_\varepsilon \left(\frac{v - m_j}{\sigma_j} \right) \right] \leq C a^{\frac{j-i}{2}},$$

where $a \in (0, 1)$ is defined in assumption **A2**. The proof of the last inequality uses the same arguments than the proof of Lemma 6. This control of covariances immediately implies (26). The tightness criterion of Kolmogorov-Chentsov applies. The proof of Corollary 1 is now complete.□

8 Proofs of the results of Subsection 5.6

In the subsequent proofs, $C > 0$ will denote a generic constant that can change from line to line. Moreover, if X is a random variable, we set $\overline{X} = X - \mathbb{E}(X)$.

We first observe that if the kernel K is Lipschitz continuous and bounded, we have for any $q \in (0, 1)$,

$$|K(x) - K(y)| \leq (2\|K\|_\infty)^{1-q} \operatorname{Lip}(K)^q |x - y|^q.$$

In particular, K is Hölder continuous with exponent q. This fact will be used in the proof of the lemmas 1 and 2.

8.1 Proof of Lemma 1

Since $\sqrt{n}\left(\hat{\theta}-\theta_0\right)=O_{\mathbb{P}}(1)$, it is enough to prove that

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\sup_{v\in I,\theta\in\Theta_{0,\epsilon}}\left|\frac{1}{\sigma_i(\theta)}K_b\left[\mathcal{L}_{v,ij}(\theta)\right]-\frac{1}{\sigma_{i\ell}(\theta)}K_b\left[\overline{\mathcal{L}}_{v,ij}(\theta)\right]\right|=o_{\mathbb{P}}\left(n^{-1/2}\right),$$

where $\epsilon>0$ is defined in assumption **A2**. Let q be a positive real number such that $2q\leq\min(\delta,s)$. Using assumption **A2**, we have

$$\max_{\theta\in\Theta}\left|\frac{1}{\sigma_i(\theta)}-\frac{1}{\overline{\sigma}_i(\theta)}\right|\leq C\max_{\theta\in\Theta}\left|\sigma_i^2(\theta)-\overline{\sigma}_i^2(\theta)\right|^q.$$

One can choose for instance $C=2^{-q}\gamma^{-1-2q}$. Moreover, setting $q=s/2$, we have

$$\begin{aligned}
& \left|K_b\left[\mathcal{L}_{v,ij}(\theta)\right]-K_b\left[\overline{\mathcal{L}}_{v,ij}(\theta)\right]\right| \\
\leq\ & Cb^{-1-q}\left|\mathcal{L}_{v,ij}(\theta)-\overline{\mathcal{L}}_{v,ij}(\theta)\right|^q \\
\leq\ & Cb^{-1-q}\left[\left|m_i(\theta)-\overline{m}_i(\theta)\right|^q+\left|m_i(\theta)\right|^q\cdot\left|\sigma_i^2(\theta)-\overline{\sigma}_i^2(\theta)\right|^q\right] \\
+\ & Cb^{-1-q}\left[\left|m_j(\theta)-\overline{m}_j(\theta)\right|^q+\left|\sigma_j^2(\theta)-\overline{\sigma}_j^2(\theta)\right|^q\cdot\left|X_j-m_j(\theta)\right|^q\right].
\end{aligned}$$

Using assumption **A2** and the Cauchy-Schwarz inequality, we deduce that

$$\sup_{v\in I}\left\|\hat{f}(v)-\widetilde{f}(v)\right\|=O_{\mathbb{P}}\left(\frac{1}{nb^{1+q}}\sum_{i=1}^{n}a^{\frac{iq}{s}}\right)=O_{\mathbb{P}}\left(\frac{1}{nb^{1+q}}\right).\Box$$

8.2 Proof of Lemma 2

Since $\sqrt{n}\left(\hat{\theta}-\theta_0\right)=O_{\mathbb{P}}(1)$, it is enough to prove that

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\sup_{v\in I,\theta\in\Theta_{0,\epsilon}}\left|\frac{1}{\sigma_i(\theta)}K_b\left[\mathcal{L}_{v,ij}(\theta)\right]-\frac{1}{\sigma_{i\ell}(\theta)}K_b\left[\mathcal{L}_{v,ij\ell}(\theta)\right]\right|=o_{\mathbb{P}}\left(n^{-1/2}\right),$$

where $\epsilon>0$ is defined in assumption **A2**.

- As in the proof of Lemma 1, we use assumption **A2** to get

$$\left|\frac{1}{\sigma_i(\theta)}-\frac{1}{\sigma_{i\ell}(\theta)}\right|\leq C\max_{\theta\in\Theta}\left|\sigma_i^2(\theta)-\sigma_{i\ell}^2(\theta)\right|^s.$$

- Using the previous point, we have also for $q=s/2$,

$$\begin{aligned}
& \sup_{v\in I}\left|K_b\left[\mathcal{L}_{v,ij}(\theta)\right]-K_b\left[\mathcal{L}_{v,ij\ell}(\theta)\right]\right| \\
\leq\ & Cb^{-1-q}\sup_{v\in I}\left|\mathcal{L}_{v,ij}(\theta)-\mathcal{L}_{v,ij\ell}(\theta)\right|^q \\
\leq\ & Cb^{-1-q}\left[\left|m_i(\theta)-m_{i\ell}(\theta)\right|^q+\left|m_i(\theta)\right|^q\cdot\left|\sigma_i^2(\theta)-\sigma_{i\ell}^2(\theta)\right|^q\right] \\
+\ & Cb^{-1-q}\left[\left|m_j(\theta)-m_{j\ell}(\theta)\right|^q+\left|X_j-X_{j\ell}\right|^q+\left|\sigma_j^2(\theta)-\sigma_{j\ell}^2(\theta)\right|^q\cdot\left|X_j-m_j(\theta)\right|^q\right].
\end{aligned}$$

Using assumption **A2**, we get

$$\mathbb{E}\sup_{\theta\in\Theta_{0,\epsilon},v\in I}\left|K_b\left[\mathcal{L}_{v,ij}(\theta)\right]-K_b\left[\mathcal{L}_{v,ij\ell}(\theta)\right]\right|\leq C\frac{a^{\ell/2}}{b^{1+q}}.$$

Using the two previous points and assumption **A2**, we get

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\sup_{v\in I,\theta\in\Theta_{0,\epsilon}}\left|\frac{1}{\sigma_i(\theta)}K_b\left[\mathcal{L}_{v,ij}(\theta)\right]-\frac{1}{\sigma_{i\ell}(\theta)}K_b\left[\mathcal{L}_{v,ij\ell}(\theta)\right]\right|=O_{\mathbb{P}}\left(\frac{a^{\ell/2}}{b^{1+q}}\right).$$

Then the result follows from the conditions $\ell\sim n^t$, $0<a<1$ and $nb\to\infty$.$\square$

8.3 Proof of Lemma 3

Let $\delta_1\in(0,2\delta/3)$ such that $t<\frac{2\delta-3\delta_1}{3(3+\delta)}$. We set $N=N_n=n^{-2/3}b^{-2-\delta_1}$. For simplicity of notations, we suppress the dependence in n of our quantities. Then we set

$$\xi_g^{(h,s,N)}=(-N)\vee\xi_g^{(h,s)}\wedge N-\mathbb{E}\left((-N)\vee\xi_g^{(h,s)}\wedge N|\mathcal{T}_{g-1}^{(s)}\right).$$

We first show that

$$\max_{h\in\mathcal{H}_n}\sum_{s=1}^{\ell}\sum_{g=1}^{k}\left|\xi_g^{(h,s)}-\xi_g^{(h,s,N)}\right|=o_{\mathbb{P}}(1). \tag{27}$$

To this end, we use the bound

$$\begin{aligned}\left|\xi_g^{(h,s)}-\xi_g^{(h,s,N)}\right| &\leq \left|\xi_g^{(h,s)}\right|\mathbb{1}_{\left|\xi_g^{(h,s)}\right|>N}+\mathbb{E}\left[\left|\xi_g^{(h,s)}\right|\mathbb{1}_{\left|\xi_g^{(h,s)}\right|>N}|\mathcal{T}_{g-1}^{(s)}\right]\\ &\leq \left|\xi_g^{(h,s)}\right|\mathbb{1}_{\left|\xi_g^{(h,s)}\right|>N}+\frac{1}{N}\mathbb{E}\left[\left|\xi_g^{(h,s)}\right|^2|\mathcal{T}_{g-1}^{(s)}\right].\end{aligned}$$

For $\epsilon>0$, we have

$$\begin{aligned}&\mathbb{P}\left(\max_{h\in\mathcal{H}}\sum_{\ell=1}^{\ell}\sum_{g=1}^{k}\left|\xi_g^{(h,s)}\right|\mathbb{1}_{\left|\xi_g^{(h,s)}\right|>N}>\epsilon\right)\\ \leq\ &\mathbb{P}\left(\max_{\substack{h\in\mathcal{H}\\1\leq s\leq\ell\\1\leq g\leq k}}\left|\xi_g^{(h,s)}\right|>N\right).\end{aligned}$$

Using the expression of N and the assumptions of the lemma, the latest probability tends to zero in probability. Moreover

$$\frac{1}{N}\max_{h\in\mathcal{H}_n}\sum_{s=1}^{\ell_n}\sum_{g=1}^{k_n}\mathbb{E}\left(\left|\xi_{n,g}^{(h,s)}\right|^2|\mathcal{T}_{n,g-1}^{(s)}\right)=O_{\mathbb{P}}\left(\frac{1}{Nnb^3}\right)=o_{\mathbb{P}}(1).$$

This show (27). To end the proof we have to show that

$$\max_{h\in\mathcal{H}}\sum_{s=1}^{\ell}\left|\sum_{g=1}^{k}\xi_g^{(h,s,N)}\right| = o_{\mathbb{P}}(1). \tag{28}$$

We will prove (28) using the exponential inequality of Freedman for martingales (see Freedman (1975)). Let $\epsilon,\epsilon'>0$. First we choose $M>0$ such that

$$\mathbb{P}\left(\max_{h\in\mathcal{H}_n}\sum_{s=1}^{\ell}\sum_{g=1}^{k}\mathbb{E}\left(\left|\xi_{n,g}^{(h,s)}\right|^2|\mathcal{T}_{n,g-1}^{(s)}\right) > \frac{M}{nb^3}\right)\le \epsilon'.$$

Using the fact that

$$\mathbb{E}\left(\left|\xi_g^{(h,s,N)}\right|^2|\mathcal{T}_{g-1}^{(s)}\right)\le \mathbb{E}\left(\left|\xi_g^{(h,s)}\right|^2|\mathcal{T}_{g-1}^{(s)}\right),$$

we get

$$\begin{aligned}
&\mathbb{P}\left(\max_{h\in\mathcal{H}}\sum_{s=1}^{\ell}\left|\sum_{g=1}^{k}\xi_g^{(h,s,N)}\right| > \epsilon,\ \max_{h\in\mathcal{H}_n}\sum_{s=1}^{\ell}\sum_{g=1}^{k}\mathbb{E}\left(\left|\xi_{n,g}^{(h,s)}\right|^2|\mathcal{T}_{g-1}^{(s)}\right)\le\frac{M}{nb^3}\right)\\
\le\ & \sum_{h\in\mathcal{H}}\sum_{s=1}^{\ell}\mathbb{P}\left(\left|\sum_{g=1}^{k}\xi_g^{(h,s,N)}\right| > \frac{\epsilon}{\ell},\sum_{g=1}^{k}\mathbb{E}\left[\left|\xi_g^{(h,s,N)}\right|^2|\mathcal{T}_{g-1}^{(s)}\right]\le\frac{M}{nb^3}\right)\\
\le\ & |2\ell\mathcal{H}|\exp\left(\frac{-\epsilon^2}{\frac{2M\ell^2}{nb^3}+4\epsilon\ell N}\right).
\end{aligned}$$

Then (28) follows from our conditions on $b,\ell,N,\mathcal{H}$. The proof of Lemma 3 is now complete.□

8.4 Proof of Lemma 4

For the proof of Lemma 4, we need two additional lemmas which are stated below. For $\theta\in\Theta_{0,\epsilon}$, the density of $\varepsilon_t(\theta)$ will be denoted by f_θ. Then, we have the expression

$$f_\theta(w)=\mathbb{E}\left[\frac{\sigma_j(\theta)}{\sigma_j(\theta_0)}f_\varepsilon\left(\frac{\sigma_j(\theta)}{\sigma_j(\theta_0)}w+\frac{m_j(\theta)-m_j(\theta_0)}{\sigma_j(\theta_0)}\right)\right]. \tag{29}$$

The following lemma is given without proof. The assertions given below are mainly a consequence of the Lebesgue theorem for the derivative of an integral depending on a parameter.

Lemma 5. *Assume that assumptions* **A3** *and* **A5** *hold true.*

1. *We have* $\sup_{\theta\in\Theta_{0,\epsilon},w\in\mathbb{R}}f_\theta(w)<\infty$. *For* $\theta\in\Theta_{0,\epsilon}$, *the function* $w\mapsto f_\theta(w)$ *has a derivative* f'_θ *such that* $\sup_{\theta\in\Theta_{0,\epsilon},w\in\mathbb{R}}|f'_\theta(w)|<\infty$. *Moreover, there exists a number* $C>0$ *not depending on* θ, w *such that*

$$\left|f'_\theta(w)-f'_{\theta_0}(w)\right|\le C\left(1+|w|\right)\|\theta-\theta_0\|.$$

2. *For each $w \in \mathbb{R}$, the function $\theta \mapsto f_\theta(w)$ is two times differentiable. Moreover, there exists a number $C > 0$ not depending on w, θ such that*

$$\left|\dot{f}_\theta(w) - \dot{f}_{\theta_0}(w)\right| \le C\left(1 + w^2\right)\|\theta - \theta_0\|, \quad \left|\dot{f}_\theta(w)\right| \le C\left(1 + |w|\right)$$

and

$$\left|\dot{f}_{\theta_0}(w_1) - \dot{f}_{\theta_0}(w_2)\right| \le C\left(1 + |w_1|\right) \cdot |w_1 - w_2|.$$

3. *There exist two constants C_1 and C_2 such that*

$$|f_\theta(w) - f_{\theta_0}(w)| \le (C_1 + C_2|w|)\,\|\theta - \theta_0\|.$$

4. *If $F : \mathbb{R} \to \mathbb{R}$ a function continuously differentiable and with a compact support. Then, for $\theta \in B(\theta_0, \epsilon)$, we have*

$$\mathbb{E}\left[F\left(\varepsilon_t(\theta)\right)\right] = \int F(w) f_\theta(w)dw, \quad \mathbb{E}\left[\dot{\varepsilon}_t(\theta)F'\left(\varepsilon_t(\theta)\right)\right] = \int F(w)\dot{f}_\theta(w)dw.$$

The next lemma is also given without proof because it results from simple computations.

Lemma 6. *Assume that assumption **A3** holds true. We set $U_j(\theta) = \frac{m_j(\theta_0) - m_j(\theta)}{\sigma_j(\theta)}$ and $V_j(\theta) = \frac{\sigma_j(\theta_0)}{\sigma_j(\theta)}$. Note that $\varepsilon_j(\theta) = U_j(\theta) + V_j(\theta)\varepsilon_j$.*

1. *We have $\mathbb{E}\left|\dot{U}_j\right|^2_{\infty,\epsilon} < \infty$, $\mathbb{E}\left|\frac{U_j\dot{V}_j}{V_j}\right|^2_{\infty,\epsilon} < \infty$, $\mathbb{E}\left|\frac{\dot{V}_j}{V_j}\right|^2_{\infty,\epsilon} < \infty$, $\mathbb{E}\left|\dot{\varepsilon}_j\right|^2_{\infty,\epsilon} < \infty$ and $\mathbb{E}\left|\ddot{\varepsilon}_j\right|_{\infty,\epsilon} < \infty$.*

2. *We have $\mathbb{E}\left[\sup_{v\in I}\left|L_{v,i}\right|^2_{\infty,\epsilon}\right] < \infty$, $\mathbb{E}\left[\sup_{v\in I}\left|\dot{L}_{v,i}\right|^2_{\infty,\epsilon}\right] < \infty$ and $\mathbb{E}\left[\sup_{v\in I}\left|\ddot{L}_{v,i}\right|_{\infty,\epsilon}\right] < \infty$.*

3. *Let η be a positive number. There exists a positive real number C such that*

$$\sup_{\substack{|v-w|\le\eta \\ \|\theta-\zeta\|\le\eta}} |L_{v,i}(\theta) - L_{w,i}(\zeta)| \le C\eta\left(1 + sup_{v\in I}\left|\dot{L}_{v,i}\right|_{\infty,\epsilon}\right)$$

and

$$\sup_{\substack{|v-w|\le\eta \\ \|\theta-\zeta\|\le\eta}} \left\|\dot{L}_{v,i}(\theta) - \dot{L}_{w,i}(\zeta)\right\| \le C\eta\left(\left|\dot{\sigma}_i, \sigma_i\right|_{\infty,\epsilon} + sup_{v\in I}\left|\ddot{L}_{v,i}\right|_{\infty,\epsilon}\right).$$

Proof of Lemma 4. We only prove the result for the conditionally heteroscedastic case, the homoscedastic uses similar arguments and is simpler.

1. It is only necessary to prove that

$$\sum_{(i,j)\in\mathcal{I}_n} \sup_{\substack{v\in I \\ \theta\in B(\theta_0,\epsilon)}} \mathbb{E}_{Y_{j\ell}}\|\dot{A}_{v,ij}(\theta)\|^2 = O_{\mathbb{P}}\left(\frac{n^2}{b^3}\right).$$

We recall that

$$\dot{A}_{v,ij}(\theta) = \dot{\sigma_{i\ell}^{-1}}(\theta) K_b \left[L_{v,i\ell}(\theta) - \varepsilon_{j\ell}(\theta)\right] + \sigma_{i\ell}^{-1}(\theta)\left[\dot{L}_{v,i\ell}(\theta) - \dot{\varepsilon}_{j\ell}(\theta)\right] K_b' \left[L_{v,i\ell}(\theta) - \varepsilon_{j\ell}(\theta)\right].$$

We define

$$U_{j\ell}(\theta) = \frac{m_{j\ell}(\theta_0) - m_{j\ell}(\theta)}{\sigma_{j\ell}(\theta)}, \quad V_{j\ell}(\theta) = \frac{\sigma_{j\ell}(\theta_0)}{\sigma_{j\ell}(\theta)}.$$

Then $\varepsilon_{j\ell}(\theta) = U_{j\ell}(\theta) + V_{j\ell}(\theta)\varepsilon_j$. Moreover

$$\begin{aligned}
& \mathbb{E}_\varepsilon \left[\|\dot{\varepsilon}_{j\ell}(\theta)\|^2 \cdot \left|K_b'\left(L_{v,i\ell}(\theta) - \varepsilon_{j\ell}(\theta)\right)\right|^2\right] \\
= & \int \frac{1}{b^3} \|\dot{U}_{j\ell}(\theta) + \frac{\dot{V}_{j\ell}(\theta)}{V_{j\ell}(\theta)}\left[L_{v,i\ell}(\theta) - U_{j\ell}(\theta) - bw\right]\|^2 \cdot \left|K'(w)\right|^2 f_\varepsilon\left(\frac{L_{v,i\ell}(\theta) - U_{j\ell}(\theta) - bw}{V_{j\ell}(\theta)}\right) dw \\
\leq & \frac{C}{b^3}\left[1 + \left|\dot{U}_{j\ell}\right|_{\infty,\epsilon}^2 + \left|\frac{\dot{V}_{j\ell}}{V_{j\ell}}\right|_{\infty,\epsilon}^2 + \sup_{v\in I}\left|L_{v,i\ell}\right|_{\infty,\epsilon}^2 \cdot \left|\frac{U_{j\ell}\dot{V}_{j\ell}}{V_{j\ell}}\right|_{\infty,\epsilon}^2\right].
\end{aligned}$$

Then we get

$$\begin{aligned}
& \mathbb{E}_{Y_{j\ell}}\left[\|\dot{A}_{v,ij}(\theta)\|^2\right] \\
\leq & \frac{C}{b^3}\left[1 + \left|\dot{\sigma}_{i\ell}^2, \sigma_{i\ell}^2\right|_{\infty,\epsilon}^2 + \sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}^2 + \mathbb{E}\left|\dot{U}_{j\ell}\right|_{\infty,\epsilon}^2 + \mathbb{E}\left|\frac{\dot{V}_{j\ell}}{V_{j\ell}}\right|_{\infty,\epsilon}^2 + \sup_{v\in I}\left|L_{v,i\ell}\right|_{\infty,\epsilon}^2 \cdot \mathbb{E}\left|\frac{U_{j\ell}\dot{V}_{j\ell}}{V_{j\ell}}\right|_{\infty,\epsilon}^2\right].
\end{aligned}$$

The result follows from assumption **A3** and Lemma 6.

2. Since

$$\dot{\varepsilon}_{j\ell}(\theta) = -\frac{\dot{m}_{j\ell}(\theta)}{\sigma_{j\ell}(\theta)} - \frac{\dot{\sigma}_{j\ell}(\theta)}{\sigma_{j\ell}(\theta)}\varepsilon_{j\ell}(\theta),$$

we have, using the compact support of the kernel K, and the equality $\frac{\dot{\sigma}_j}{\sigma_j} = \frac{\dot{\sigma}_j^2}{2\sigma_j^2}$,

$$\|\dot{\varepsilon}_{j\ell}(\theta)\| \cdot \left|K_b'\left(L_{v,i\ell}(\theta) - \varepsilon_{j\ell}(\theta)\right)\right| \leq \frac{C}{b^2}\left[\left|\dot{m}_{j\ell}, \sigma_{j\ell}\right|_{\infty,\epsilon} + \left|\dot{\sigma}_{j\ell}^2, \sigma_{j\ell}^2\right|_{\infty,\epsilon} \cdot \left[1 + \sup_{v\in I}\left|L_{v,i\ell}\right|_{\infty,\epsilon}\right]\right].$$

Then we conclude that

$$\begin{aligned}
& \sup_{v\in I}\left\|\dot{A}_{v,ij}\right\| \\
\leq & \frac{C}{b^2}\left[\left|\dot{\sigma}_{i\ell}^2, \sigma_{i\ell}^2\right|_{\infty,\epsilon} + \sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon} + \left|\dot{m}_{j\ell}, \sigma_{j\ell}\right|_{\infty,\epsilon} + \left|\dot{\sigma}_{j\ell}^2, \sigma_{j\ell}^2\right|_{\infty,\epsilon} \cdot \left[1 + \sup_{v\in I}\left|L_{v,i\ell}\right|_{\infty,\epsilon}\right]\right].
\end{aligned}$$

From the assumption **A3**, we have $\max_{1\leq j\leq n}\left|\dot{\sigma}_{j\ell}^2, \sigma_{j\ell}^2\right|_{\infty,\epsilon} = O_{\mathbb{P}}\left(n^{1/3}\right)$ and $\max_{1\leq j\leq n}\left|\dot{m}_{j\ell}, \sigma_{j\ell}\right|_{\infty,\epsilon} = O_{\mathbb{P}}\left(n^{1/3}\right)$. Then the result follows from the point 2 of Lemma 6.

3. If $(v,w)\in I^2$ and $(\theta,\zeta)\in\Theta_{0,\epsilon}$ are such that $|v-w|\leq\eta$ and $\|\theta-\zeta\|\leq\eta$, some basic computations lead to the inequality

$$\begin{aligned}\left|\dot{A}_{v,ij}(\theta)-\dot{A}_{w,ij}(\zeta)\right| &\leq \frac{C\eta}{b^3}\left[1+\left|\ddot{\sigma}^2_{i\ell},\sigma^2_{i\ell}\right|_{\infty,\epsilon}+\left|\dot{\sigma}^2_{i\ell},\sigma^2_{i\ell}\right|^2_{\infty,\epsilon}\right]\\ &+\frac{C\eta}{b^3}\left[\sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|^2_{\infty,\epsilon}+|\dot{\varepsilon}_{j\ell}|^2_{\infty,\epsilon}+\sup_{v\in I}\left|\ddot{L}_{v,i\ell}\right|_{\infty,\epsilon}+|\ddot{\varepsilon}_{j\ell}|_{\infty,\epsilon}\right].\end{aligned}$$

Then the result follows from assumption **A3** and Lemma 6.□

9 Proofs of assertions (18) and (19) in the proof of Theorem 2

9.1 Proof of assertion (18)

The following notations will be needed. We define

$$\Gamma_i^{(1)}(v,\theta)=\dot{\sigma}^{-1}_{i\ell}(\theta)\int K(w)f_\theta\left(L_{v,i\ell}(\theta)-bw\right)dw,$$

$$\Gamma_i^{(2)}(v,\theta)=\frac{\dot{L}_{v,i\ell}(\theta)}{\sigma_{i\ell}(\theta)}\int K(w)f'_\theta\left(L_{v,i\ell}(\theta)-bw\right)dw,$$

$$\Gamma_i^{(3)}(v,\theta)=\frac{1}{\sigma_{i\ell}(\theta)}\int K(w)\dot{f}_\theta\left(L_{v,i\ell}(\theta)-bw\right)dw.$$

Note that from Lemma 5, 4., we have

$$\mathbb{E}_{Y_{j\ell}}\dot{A}_{v,ij}(\theta)=\sum_{h=1}^{3}\Gamma_i^{(h)}(v,\theta).$$

Then assertion (21) will follow if we show that for $h=1,2,3$,

$$\frac{1}{n^2}\sum_{(i,j)\in\mathcal{I}_n}\sup_{v\in I,\theta\in\Theta_{0,n}}\left\|\Gamma_i^{(h)}(v,\theta)-\Gamma_i^{(h)}(v,\theta_0)\right\|=o_{\mathbb{P}}(1).\tag{30}$$

The proof of (30) follows from the following bounds, Assumption **A3**, Lemma 5 and Lemma 6.

- For $h=1$, we have

$$\begin{aligned}&\sup_{v\in I,\theta\in\Theta_{0,n}}\left\|\Gamma_i^{(1)}(v,\theta)-\Gamma_i^{(1)}(v,\theta_0)\right\|\\ \leq\ &\frac{C}{\sqrt{n}}\left[\left|\ddot{\sigma}^{-1}_{i\ell}\right|_{\infty,\epsilon}+\left|\dot{\sigma}^{-1}_{i\ell}\right|_{\infty,\epsilon}\left(\sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}+C_1+C_2\left(\sup_{v\in I}|L_{v,i\ell}|_{\infty,\epsilon}+1\right)\right)\right],\end{aligned}$$

  where C_1 and C_2 are the constants given in Lemma 5 (3).

- For $h = 2$, we have

$$\sup_{v\in I,\theta\in\Theta_{0,n}} \left\|\Gamma_i^{(2)}(v,\theta) - \Gamma_i^{(2)}(v,\theta_0)\right\| \leq \frac{C}{\sqrt{n}}\left[\sup_{v\in I}\left|\ddot{L}_{v,i\ell}\right|_{\infty,\epsilon} + \sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}\cdot|\dot{\sigma}_{i\ell},\sigma_{i\ell}|_{\infty,\epsilon}\right] + \frac{C}{\sqrt{n}}\sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}\cdot\left(1+\sup_{v\in I}|L_{v,i\ell}|_{\infty,\epsilon} + \sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}\right).$$

- Finally we have

$$\sup_{v\in I,\theta\in\Theta_{0,n}} \left\|\Gamma_i^{(3)}(v,\theta) - \Gamma_i^{(3)}(v,\theta_0)\right\| \leq \frac{C}{\sqrt{n}}\left[|\dot{\sigma}_{i\ell},\sigma_{i\ell}|_{\infty,\epsilon}\left(1+\sup_{v\in I}|L_{v,i\ell}|_{\infty,\epsilon}\right)\right] + \frac{C}{\sqrt{n}}\left[1+\sup_{v\in I}|L_{v,i\ell}|^2_{\infty,\epsilon} + \sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}\cdot\left(1+\sup_{v\in I}|L_{v,i\ell}|_{\infty,\epsilon}\right)\right].$$

Using the integrability properties stated in Lemma 6 and assumption **A3**, assertion (18) given in the paper follows.

9.1.1 Proof of assertion (19)

We set

$$\Delta_{v,i}^{(1)} = \frac{-\dot{\sigma}_{i\ell}(\theta_0)}{\sigma_{i\ell}^2(\theta_0)} f_{\theta_0}\left(L_{v,i\ell}(\theta_0)\right), \quad \Delta_{v,i}^{(2)} = \frac{\dot{L}_{v,i\ell}(\theta_0)}{\sigma_{i\ell}(\theta_0)} f'_{\theta_0}\left(L_{v,i\ell}(\theta_0)\right), \quad \Delta_{v,i}^{(3)} = \frac{1}{\sigma_{i\ell}(\theta_0)}\dot{f}_{\theta_0}\left(L_{v,i\ell}(\theta_0)\right).$$

Using Lemma 5 and Lemma 6, it is easily seen that for $h = 1,2,3$,

$$\frac{1}{n^2}\sum_{(i,j)\in\mathcal{I}_n}\sup_{v\in I}\|\Gamma_{v,i}(\theta_0) - \Delta_{v,i}\| = o_{\mathbb{P}}(1).$$

To end the proof of assertion (19), it remains to show that for $h = 1,2,3$,

$$\sup_{v\in I}\frac{1}{n^2}\left|\sum_{(i,j)\in\mathcal{I}_n}\left[\Delta_{v,i}^{(h)} - \mathbb{E}\left(\Delta_{v,i}^{(h)}\right)\right]\right| = o_{\mathbb{P}}(1). \tag{31}$$

Note that $\sum_{h=1}^3 \mathbb{E}\left(\Delta_{v,i}^{(h)}\right) = \dot{h}_{\theta_0}(v)$. To show (31), we first notice that $\sup_{n\in\mathbb{N}^*,v\in I}\mathbb{E}\|\Delta_{v,i}^{(h)}\| < \infty$ for $h = 1,2,3$. Then, since $\ell = o(n)$, it is easily seen that

$$\sup_{v\in I}\left|\frac{1}{n^2}\sum_{(i,j)\in\mathcal{I}_n}\left[\Delta_{v,i}^{(h)} - \mathbb{E}\left(\Delta_{v,i}^{(h)}\right)\right] - \frac{1}{n}\sum_{i=1}^{n'}\left[\Delta_{v,i}^{(h)} - \mathbb{E}\left(\Delta_{v,i}^{(h)}\right)\right]\right| = o_{\mathbb{P}}(1).$$

Now, we set for $v \in I$ and $h = 1,2,3$,

$$G_n^{(h)}(v) = \frac{1}{n}\sum_{i=1}^{n'}\left[\Delta_{v,i}^{(h)} - \mathbb{E}\left(\Delta_{v,i}^{(h)}\right)\right].$$

Then assertion (19) will follow if we show that

$$\sup_{v\in I}\left\|G_n^{(h)}(v)\right\| = o_{\mathbb{P}}(1), \quad h = 1,2,3. \tag{32}$$

We have using the $\ell-$dependence,

$$\mathbb{E}\left|G_n^{(h)}(v)\right|^2 \leq \frac{\ell}{n^2}\sum_{s=1}^{\ell}\sum_{g=0}^{k-1}\mathrm{Var}\left(M_{s+g\ell}^{(h)}(h)\right).$$

Moreover, using the fact that $\ell = o(n)$ and that $\sup_{n,i\geq 1}\sup_{v\in I}\mathrm{Var}\left(M_i^{(h)}(v)\right)$ is bounded, we get $G_n^{(h)}(v) = o_{\mathbb{P}}(1)$ for each $v \in I$. Now, (32) will follow if we show that $\sup_{v\neq \bar{v}}\frac{\|G_n^{(h)}(v)-G_n^{(h)}(\bar{v})\|}{|v-\bar{v}|} = O_{\mathbb{P}}(1)$. But this is a consequence of the following bounds. First, there exists $C > 0$ such that

$$\left\|\Delta_{v,i}^{(1)} - \Delta_{v',i}^{(1)}\right\| \leq C\left|\dot{\sigma}_{i\ell},\sigma_{i\ell}\right|_{\infty,\epsilon}|v-v'|,$$

$$\left\|\Delta_{v,i}^{(2)} - \Delta_{v',i}^{(2)}\right\| \leq C\left[\left|\dot{\sigma}_{i\ell},\sigma_{i\ell}\right|_{\infty,\epsilon} + \sup_{v\in I}\left|\dot{L}_{v,i\ell}\right|_{\infty,\epsilon}\right]\cdot|v-v'|.$$

For $h = 3$, we set $E_1 = \mathbb{E}\left[\frac{\dot{\sigma}_j(\theta_0)}{\sigma_j(\theta_0)}\right]$ and $E_2 = \mathbb{E}\left[\frac{\dot{m}_j(\theta_0)}{\sigma_j(\theta_0)}\right]$. Then $\dot{f}_{\theta_0}(w) = E_1\left(f_\varepsilon(w) + wf_\varepsilon'(w)\right) + E_2 f_\varepsilon'(w)$. Then, using assumption **A5**, we have

$$\begin{aligned}
&\left\|\Delta_{v,i}^{(3)} - \Delta_{v',i}^{(3)}\right\| \\
\leq\ & C\left[|v-v'| + \left|L_{v,i\ell}(\theta_0)f_\varepsilon'\left(L_{v,i\ell}(\theta_0)\right) - L_{v',i\ell}(\theta_0)f_\varepsilon'\left(L_{v',i\ell}(\theta_0)\right)\right|\right] \\
\leq\ & C|v-v'|\cdot\left[1+\sup_{v\in I}\left|L_{v,i\ell}\right|_{\infty,\epsilon}\right].
\end{aligned}$$

In the previous bounds, the real number C does not depends on $v,\bar{v}\in I$. Then (32) follows and the proof of assertion (19) is now complete.

10 Checking the regularity assumptions on densities

10.1 Density regularities of ARCH processes

Here, we assume that (X_t) is a stationary ARCH process defined by

$$X_t = \varepsilon_t\sigma_t, \quad \sigma_t^2 = \alpha_0 + \sum_{j\geq 1}\alpha_j X_{t-j}^2.$$

We assume here that $\sum_{j=1}^\infty \alpha_j < \infty$ and that f_ε is bounded. We set $\mu(dx) = \sup_{|z|\leq z_0} f_\varepsilon(x+z)dx$. We denote by f_{σ^2} the probability density of the conditional variance σ_t^2 and for $v\neq 0$, $g_v(x) = \frac{2v}{x^2}f_{\sigma^2}\left(\frac{v^2}{x^2}\right)$ which is well defined for $x\neq 0$. We also set $G(x) = \sup_{v\in I} g_v(x)$ and for a compact interval I which does not contain 0, $\mathcal{G}_I = \{g_v : v\in I\}$.

Lemma 7. *1. Assume that $\alpha_1,\alpha_2 > 0$. Then f_{σ^2} is bounded. Moreover, if $\mathbb{E}\sigma_t < \infty$, then for all $v\neq 0$, we have $\int g_v(x)^2\mu(dx) < \infty$.*

2. *Assume that* $\alpha_1, \alpha_2, \alpha_3 > 0$. *Then there exists a constant* $C > 0$ *such that for* $s_1, s_2 \geq 0$*, we have*
$$|f_{\sigma^2}(s_2) - f_{\sigma^2}(s_1)| \leq C\left(1 + \sqrt{|s_1|} \wedge \sqrt{|s_2|}\right) \cdot \sqrt{|s_2 - s_1|}.$$

3. *In addition to the previous point, assume that there exists* $\delta \in (0, \frac{1}{2})$ *such that* $\mathbb{E}\sigma_t^{\frac{3}{2}+\delta} < \infty$ *and that* $x \mapsto |x|^{\frac{3}{2}+\delta} f_\varepsilon(x)$ *is bounded. Then there exists a real number* $o > 0$ *such that* $\int G(x)^{2+o}\mu(dx) < \infty$. *Moreover there exists some constants* $\zeta, C > 0$ *such that*
$$N_{[]}\left(\epsilon, \mathcal{G}_I, \mathbb{L}^2(\mu)\right) \leq C\epsilon^{-\zeta}.$$

Proof of Lemma 7. Before proving the lemma, we first derive an expression for f_{σ^2} involving conditional distributions. We will use for $j \geq 1$ the notation $\mathbf{z_j} = (z_j, z_{j+1}, \ldots)$, we set We set $k_3(\mathbf{z_3}) = \alpha_0 + \sum_{j=3}^{\infty} \alpha_j z_j^2$ and
$$s(\mathbf{z_1}) = \sqrt{\alpha_0 + \sum_{j=1}^{\infty} \alpha_j z_j^2}.$$
The measure ν will denote the probability distribution of $(X_{t-1}, X_{t-2}, \ldots)$. Moreover, we set
$$r(z_1, z_2|\mathbf{z_3}) = \frac{1}{s(\mathbf{z_2})} f_\varepsilon\left(\frac{z_1}{s(\mathbf{z_2})}\right) \frac{1}{s(\mathbf{z_3})} f_\varepsilon\left(\frac{z_2}{s(\mathbf{z_3})}\right)$$
and
$$\bar{r}(z_1, z_2|\mathbf{z_3}) = \frac{1}{\sqrt{\alpha_1 \cdot \alpha_2}} r\left(\frac{z_1}{\sqrt{\alpha_1}}, \frac{z_2}{\sqrt{\alpha_2}}|\mathbf{z_3}\right).$$
If $h : \mathbb{R} \to \mathbb{R}$ is a bounded and measurable function, we have
$$\begin{aligned}
\mathbb{E}h\left(\sigma_t^2\right) &= \int\int\int h\left(\alpha_1 z_1^2 + \alpha_2 z_2^2 + k_3(\mathbf{z_3})\right) r(z_1, z_2|\mathbf{z_3}) dz_1 dz_2 d\nu(\mathbf{z_3}) \\
&= \int\int\int h\left(z_1^2 + z_2^2 + k_3(\mathbf{z_3})\right) \bar{r}(z_1, z_2|\mathbf{z_3}) dz_1 dz_2 d\nu(\mathbf{z_3}) \\
&= \int_0^\infty \int_{-\pi}^{\pi} \int h\left(\rho^2 + k_3(\mathbf{z_3})\right) \bar{r}(\rho\cos(\phi), \rho\sin(\phi)|\mathbf{z_3}) \rho d\rho d\phi d\nu(\mathbf{z_3}) \\
&= \int\int_{k_3(\mathbf{z_3}) \leq s} \int_{-\pi}^{\pi} h(s) \frac{1}{2} \bar{r}(\sqrt{s - k_3(\mathbf{z_3})}\cos(\phi), \sqrt{s - k_3(\mathbf{z_3})}\sin(\phi)|\mathbf{z_3}) ds d\phi d\nu(\mathbf{z_3}).
\end{aligned}$$

Then we deduce that for $s \geq 0$,
$$f_{\sigma^2}(s) = \frac{1}{2}\int_{k_3(\mathbf{z_3}) \leq s} \int_{-\pi}^{\pi} \bar{r}(\sqrt{s - k_3(\mathbf{z_3})}\cos(\phi), \sqrt{s - k_3(\mathbf{z_3})}\sin(\phi)|\mathbf{z_3}) d\phi d\nu(\mathbf{z_3}). \tag{33}$$

1. The fact that f_{σ^2} is bounded is a consequence of the expression (33), using the fact that f_ε and then $\bar{r}$ are bounded. Moreover, we have
$$\begin{aligned}
\int g_v(x)^2 \mu(dx) &\leq 4v^2 \cdot \|f_\varepsilon\|_\infty \int \frac{1}{x^4} f_{\sigma^2}\left(\frac{v^2}{x^2}\right) dx \\
&\leq \frac{4}{|v|} 4\|f_\varepsilon\|_\infty \int_0^\infty \sqrt{y} f_{\sigma^2}(y) dy.
\end{aligned}$$
This bound gives the result.

2. We use the expression (33). Using some basic computations, it is easily seen that for real numbers z_1, z_2,

$$|r(z_1, z_2|\mathbf{z_3}) - r(\bar{z}_1, \bar{z}_2|\mathbf{z_3})| \leq C\left(|z_1 - \bar{z}_1| + (1 + |z_1| \wedge |\bar{z}_1|) \cdot |z_2 - \bar{z}_2|\right)$$

for some constant $C > 0$. Setting now for $s \geq k_3(\mathbf{z_3})$,

$$h(s, \phi, \mathbf{z_3}) = \bar{r}\left[\sqrt{s - k_3(\mathbf{z_3})}\cos(\phi), \sqrt{s - k_3(\mathbf{z_3})}\sin(\phi)|\mathbf{z_3}\right],$$

we have for $s_2 \geq s_1 \geq k_3(\mathbf{z_3})$,

$$|h(s_1, \phi, \mathbf{z_3}) - h(s_2, \phi, \mathbf{z_3})| \leq C\left(1 + \sqrt{s_1 \wedge s_2}\right) \cdot \sqrt{|s_2 - s_1|}.$$

We deduce that for $s_1, s_2 \geq 0$,

$$|f_{\sigma^2}(s_2) - f_{\sigma^2}(s_1)| \leq C\left(1 + \sqrt{s_1 \wedge s_2}\right) \cdot \sqrt{|s_2 - s_1|} + \mathbb{P}\left(s_1 < k_3(X_{t-3}, \ldots) \leq s_2\right).$$

Moreover, it is easily seen that $k_3\left(X_{t-3}, \ldots\right)$ has a density q_k such that

$$q_k(x) \leq C \int_{k_4(\mathbf{z_4}) \leq x} \left(x - k_4(\mathbf{z_4})\right)^{-1/2} d\nu(\mathbf{z_4}),$$

where $k_4(\mathbf{z_4}) = \alpha_0 + \sum_{j \geq 4} \alpha_j z_j^4$. Then, it can be shown that

$$\mathbb{P}\left(s_1 < k_3(X_{t-3}, \ldots) \leq s_2\right) \leq C\sqrt{|s_2 - s_1|}.$$

This proves the second point of this lemma.

3. We first show that under our assumptions,

$$\sup_{x>0} x^{\frac{3}{4}+\frac{\delta}{2}} f_{\sigma^2}(x) < \infty. \tag{34}$$

We first observe that

$$\begin{aligned}
& |u|^{\frac{3}{2}+\delta}\bar{r}\left(u\cos\phi, u\sin\phi|\mathbf{z_3}\right) \\
\leq\ & C\left[|u\cos\phi|^{\frac{3}{2}+\delta} + |u\sin\phi|^{\frac{3}{2}+\delta}\right] \cdot \bar{r}\left(u\cos\phi, u\sin\phi|\mathbf{z_3}\right) \\
\leq\ & C\sup_{y>0}\left\{y^{\frac{3}{2}+\delta} f_\varepsilon(y)\right\} \cdot \frac{\left|\alpha_0 + \alpha_1 u^2\sin^2\phi + \sum_{j\geq 2}\alpha_j z_{j+1}^2\right|^{\frac{1}{4}+\frac{\delta}{2}}}{s(\mathbf{z_3})^{\frac{1}{2}+\delta}} f_\varepsilon\left(\frac{u\sin\phi}{\sqrt{\alpha_2}s(\mathbf{z_3})}\right) \\
+\ & C\sup_{y>0}\left\{y^{\frac{3}{2}+\delta} f_\varepsilon(y)\right\} \cdot s(\mathbf{z_3})^{\frac{1}{2}+\delta} \\
\leq\ & C\sup_{y>0}\left\{y^{\frac{3}{2}+\delta} f_\varepsilon(y)\right\} \cdot \left[\sup_{y>0}\left\{y^{\frac{1}{2}+\delta} f_\varepsilon(y)\right\} + \left|\alpha_0 + \sum_{j\geq 2}\alpha_j z_{j+1}^2\right|^{\frac{1}{4}+\frac{\delta}{2}} + s(\mathbf{z_3})^{\frac{1}{2}+\delta}\right]
\end{aligned}$$

Then we deduce that the function ϑ defined by

$$\vartheta(\mathbf{z_3}) = \sup_{u\in\mathbb{R},\phi\in(-\pi,\pi)} |u|^{\frac{3}{2}+\delta}\bar{r}\left(u\cos\phi, u\sin\phi|\mathbf{z_3}\right)$$

satisfies $\mathbb{E}\left[\vartheta\left(X_{t-3},X_{t-4},\ldots\right)\right]<\infty$. Using the expression (33), condition $\mathbb{E}\sigma_t^{\frac{3}{2}+\delta}<\infty$ and the decomposition $x=x-k_3(\mathbf{z_3})+k_3(\mathbf{z_3})$, we get the bound

$$\begin{aligned}
& x^{\frac{3}{4}+\frac{\delta}{2}}f_{\sigma^2}(x) \\
\leq\ & C\left(\mathbb{E}\left[k_3(X_{t-3},X_{t-4},\ldots)^{\frac{3}{4}+\frac{\delta}{2}}\right]+\mathbb{E}\left[\vartheta\left(X_{t-3},X_{t-4},\ldots\right)\right]\right) \\
<\ & \infty.
\end{aligned}$$

This shows (34). For simplicity, we now assume that $I\subset(0,\infty)$ (the case $I\subset(-\infty,0)$ is identical). First, we show that there exists $o>0$ such that $\int G(x)^{2+o}\mu(dx)<\infty$. We have Since I is compact and $\sigma_t^2(\theta_0)$ is bounded from below, there exists $\varpi>0$ such that $G(x)=0$ when $x>\varpi$. We choose o such that $\frac{3}{4}+\frac{\delta}{2}=1-\frac{1-o}{2(2+o)}$. We get

$$\begin{aligned}
\int_0^{\varpi} G(x)^{2+o}dx &\leq C\int_0^{\varpi}\frac{1}{x^{1-o}}\left|\frac{x^{\frac{1-o}{2+o}}}{x^2}\sup_{v\in I}f_{\sigma^2}\left(\frac{v^2}{x^2}\right)\right|^{2+o}dx \\
&\leq C\left[\sup_{y>0}\left\{y^{\frac{3}{4}+\frac{\delta}{2}}f_{\sigma^2}(y)\right\}\right]^{2+o}.
\end{aligned}$$

This shows that $\int_0^\infty G(x)^{2+o}dx<\infty$. Finally, we consider the bracketing numbers of the family $\mathcal{G}_I$. Let $\eta\in(0,1)$ be such that $8\eta+\left(\frac{1}{2}-\delta\right)(2-2\eta)<1$. Let $v_1,v_2\in I$. We have

$$\begin{aligned}
|g_{v_1}(x)-g_{v_2}(x)| &\leq CG(x)|v_1-v_2|+\frac{2v}{x^2}\left|f_{\sigma^2}\left(\frac{v_1^2}{x^2}\right)-f_{\sigma^2}\left(\frac{v_2^2}{x^2}\right)\right| \\
&\leq C\left[G(x)|v_1-v_2|+\frac{G(x)^{1-\eta}}{x^{2\eta}}\left|f_{\sigma^2}\left(\frac{v_1^2}{x^2}\right)-f_{\sigma^2}\left(\frac{v_2^2}{x^2}\right)\right|^{\eta}\right] \\
&\leq C\left[G(x)+G(x)^{1-\eta}\left(|x|^{-3\eta}+x^{-4\eta}\right)\right]\cdot|v_1-v_2|^{\eta}.
\end{aligned}$$

Since we have

$$\int_0^{\varpi}G(x)^{2+o}dx<\infty,\quad \int_0^{\varpi}G(x)^{2-2\eta}x^{-8\eta}dx<\infty,$$

where the second integrability condition follows from the assumption on η and (34), the bound given for $N_{[]}\left(\epsilon,\mathcal{G}_I,\mathbb{L}^2(\mu)\right)$ easily follows (see van der Vaart (1998), Example 19.7, for $\eta=1$ and μ a probability measure, but the arguments are similar in our case). This completes the proof of the lemma.□

10.2 A result for ARMA-GARCH processes

Lemma 8. *Assume that $(\psi_j)_{j\geq1}$ and $(\alpha_j)_{j\geq0}$ are two summable sequences of real numbers such that $\alpha_i\geq0$ for $i\geq0$ and $\alpha_i>0$ for $0\leq i\leq3$ and there exists an integer $q\geq1$ such that $\psi_q\neq0$. Let $(Z_t)_{t\in\mathbb{Z}}$ be a stationary process of real random variables such that $\mathbb{E}Z_t^2<\infty$ and such that the conditional distribution of $Z_t|Z_{t-1},Z_{t-2},\ldots$ has a bounded density. Then the density ω of the couple $\left(\sum_{j=1}^\infty\psi_jZ_{t-j},\sqrt{\alpha_0+\sum_{j=1}^\infty\alpha_jZ_{t-j}^2}\right)$ satisfies $\omega(x,y)\leq Cy$ for a positive constant C.*

Proof of Lemma 8. We set $\mathbf{z_i} = (z_i, z_{i+1}, \ldots)$ and we denote by $\widetilde{f}(\cdot|\mathbf{z_1})$ the conditional density of Z_t given that $Z_{t-i} = z_i$ for $i \geq 1$. We consider two cases.

1. We first assume that $q \leq 3$. We set

$$k_1(\mathbf{z_4}) = \sum_{j\geq 4} \psi_j z_j, \quad k_2(\mathbf{z_4}) = \alpha_0 + \sum_{j\geq 4} \alpha_j z_j^2$$

and $r = \|a\|$ with $a = \left(\frac{\psi_i}{\sqrt{\alpha_i}}\right)_{1\leq i\leq 3}$. We also set

$$\zeta(\mathbf{z_1}) = \frac{1}{\sqrt{\alpha_1\alpha_2\alpha_3}} \widetilde{f}\left(\frac{z_1}{\sqrt{\alpha_1}}|\frac{z_2}{\sqrt{\alpha_2}}, \frac{z_3}{\sqrt{\alpha_3}}, \mathbf{z_4}\right) \cdot \widetilde{f}\left(\frac{z_2}{\sqrt{\alpha_2}}|\frac{z_3}{\sqrt{\alpha_3}}, \mathbf{z_3}\right) \cdot \widetilde{f}\left(\frac{z_3}{\sqrt{\alpha_3}}|\mathbf{z_4}\right).$$

We also denote by $\mathcal{R}$ the rotation of $\mathbb{R}^3$ such that $\mathcal{R}e_1 = a/r$ where $e_1 = (1, 0, 0)$. For simplicity of notations, we only use one sign "integral" and do not precise the boundaries for integration in the next computations. Then if ν denotes the probability distribution of $(Z_{t-4}, Z_{t-5}, \ldots)$, we have

$$\begin{aligned}
&\mathbb{E}h\left(\sum_{j=1}^{\infty} \psi_j Z_{t-j}, \sqrt{\alpha_0 + \sum_{j=1}^{\infty} \alpha_j Z_{t-j}^2}\right) \\
= &\int h\left(\sum_{j=1}^{3} a_j z_j + k_1(\mathbf{z_4}), \sqrt{\sum_{j=1}^{3} z_j^2 + k_2(\mathbf{z_4})}\right) \zeta(\mathbf{z_1}) dz_1 dz_2 dz_3 d\nu(\mathbf{z_4}) \\
= &\int h\left(rz_1 + k_1(\mathbf{z_4}), \sqrt{\sum_{j=1}^{3} z_j^2 + k_2(\mathbf{z_4})}\right) \zeta\left(\mathcal{R}(z_1, z_2, z_3), \mathbf{z_4}\right) dz_1 dz_2 dz_2 d\nu(\mathbf{z_4}) \\
= &\int h\left(rz_1 + k_1(\mathbf{z_4}), \sqrt{z_1^2 + \rho^2 + k_2(\mathbf{z_4})}\right) \zeta\left(\mathcal{R}(z_1, \rho\cos\phi, \rho\sin\phi), \mathbf{z_4}\right) \rho dz_1 d\rho d\phi d\nu(\mathbf{z_4}) \\
= &\int h(x, y) \\
&\zeta\left(\mathcal{R}\left(\frac{x - k_1(\mathbf{z_4})}{\alpha}, \sqrt{y^2 - \left[\frac{x - k_1(\mathbf{z_4})}{\alpha}\right] - k_2(\mathbf{z_4})}\cos\phi, \sqrt{y^2 - \left[\frac{x - k_1(\mathbf{z_4})}{\alpha}\right] - k_2(\mathbf{z_4})}\sin\phi\right), \mathbf{z_4}\right) \\
&\frac{y}{r} dx dy d\phi d\nu(\mathbf{z_4}).
\end{aligned}$$

Since ζ is bounded, it is easily seen that $\omega(x, y) \leq Cy$ for a positive constant C.

2. We next consider the case $q \geq 4$. We set

$$\zeta_1(\mathbf{z_1}) = \prod_{i=1}^{2} \widetilde{f}\left(z_i|\mathbf{z_{i+1}}\right), \quad \zeta_2(\mathbf{z_3}) = \prod_{i=3}^{q-1} \widetilde{f}\left(z_i|\mathbf{z_{i+1}}\right)$$

and $k_3(\mathbf{z_3}) = \alpha_0 + \sum_{j=3}^{\infty} \alpha_j z_j^2$. For simplicity of notations, we assume that $\alpha_1 = \alpha_2 = 1$ (otherwise, as in the previous point, a change of variables is needed in the computations given below). Denoting by ν, the probability distribution of $(Z_{q+1}, Z_{q+2}, \ldots)$, we have

$$
\begin{aligned}
& \mathbb{E}h\left(\sum_{j=1}^{\infty}\psi_j Z_{t-j}, \sqrt{\alpha_0+\sum_{j=1}^{\infty}\alpha_j Z_{t-j}^2}\right) \\
= \; & \int h\left(\sum_{j=q}^{\infty}\psi_j z_j, \sqrt{z_1^2+z_2^2+k_3(\mathbf{z_3})}\right)\prod_{i=1}^{q}\widetilde{f}(z_i|\mathbf{z_{i+1}})dz_1\cdots dz_q d\nu(\mathbf{z_{q+1}}) \\
= \; & \int h\left(\sum_{j=q}^{\infty}\psi_j z_j, \sqrt{\rho^2+k_3(\mathbf{z_3})}\right)\zeta_1(\rho\cos\phi, \rho\sin\phi, \mathbf{z_3})\zeta_2(\mathbf{z_3}) \\
& \widetilde{f}(z_q|\mathbf{z_{q+1}})\rho d\rho d\phi dz_3\cdots dz_q d\nu(\mathbf{z_{q+1}}) \\
= \; & \int h\left(\sum_{j=q}^{\infty}\psi_j z_j, y\right)\zeta_1\left(\sqrt{y^2-k_3(\mathbf{z_3})}\cos\phi, \sqrt{y^2-k_3(\mathbf{z_3})}\cos\phi, \mathbf{z_3}\right) \\
& \zeta_2(\mathbf{z_3})\widetilde{f}(z_q|\mathbf{z_{q+1}})ydyd\phi dz_3\cdots dz_q d\nu(\mathbf{z_{q+1}}) \\
= \; & \int h(x,y)\zeta_1\left(\sqrt{y^2-k_3(\mathbf{z_3})}\cos\phi, \sqrt{y^2-k_3(\mathbf{z_3})}\cos\phi, z_3,\ldots,z_{q-1}, \frac{x-\sum_{j\geq q+1}\psi_j z_j}{\psi_q}, \mathbf{z_{q+1}}\right) \\
& \zeta_2\left(z_3,\ldots,z_{q-1}, \frac{x-\sum_{j\geq q+1}\psi_j z_j}{\psi_q}, \mathbf{z_{q+1}}\right)\widetilde{f}\left(\frac{x-\sum_{j\geq q+1}\psi_j z_j}{\psi_q}|\mathbf{z_{q+1}}\right) \\
& ydxdyd\phi dz_3\cdots dz_{q-1}d\nu(\mathbf{z_{q+1}})
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\omega(x,y) \qquad & = y\int\zeta_1\left(\sqrt{y^2-k_3(\mathbf{z_3})}\cos\phi, \sqrt{y^2-k_3(\mathbf{z_3})}\cos\phi, z_3,\ldots,z_{q-1}, \frac{x-\sum_{j\geq q+1}\psi_j z_j}{\psi_q}, \mathbf{z_{q+1}}\right) \\
& \zeta_2\left(z_3,\ldots,z_{q-1}, \frac{x-\sum_{j\geq q+1}\psi_j z_j}{\psi_q}, \mathbf{z_{q+1}}\right) \\
& \widetilde{f}\left(\frac{x-\sum_{j\geq q+1}\psi_j z_j}{\psi_q}|\mathbf{z_{q+1}}\right)d\phi dz_3\cdots dz_{q-1}d\nu(\mathbf{z_{q+1}}).
\end{aligned}
$$

We deduce that there exists $C>0$ such that $\omega(x,y)\leq Cy$. $\square$

11 Adequation tests

11.1 Proof of Corollary 2

We first apply Theorem 2.2 in Liu and Wu (2010). The assumptions $(C1)$ and $(C4)$ used in that paper for the bandwidth and the kernel are automatically satisfied here. We then check $(C3)'$. Since the noise density f_ε is positive, the marginal density f_X is also positive. The conditional density of $X_t|X_{t-1}, X_{t-2}...$ is given by $v\mapsto\frac{1}{\sigma_t(\theta_0)}f_\varepsilon\left(\frac{v-m_t(\theta_0)}{\sigma_t(\theta_0)}\right)$. Using the condition

$$\sup_{x\in\mathbb{R}}\left[|f_\varepsilon(x)|+\left|f_\varepsilon'(x)\right|+\left|f_\varepsilon''(x)\right|\right]<\infty,$$

we easily deduce the boundedness of the conditional distribution and its two first derivatives and then $(C3)'$. Finally, $(C2)'$ is a straightforward consequence of our assumption **A2**. Indeed, there exists a measurable function $H : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that $X_t = H(\varepsilon_t, \varepsilon_{t-1}, \ldots)$. Moreover, from Assumption **A2**, we have, setting $X_{t\ell} = H\left(\varepsilon_t, \ldots, \varepsilon_{t-\ell+1}, \varepsilon'_{t-\ell}, \varepsilon'_{t-\ell-1}, \ldots\right)$,

$$\mathbb{E}\left|X_t - X_{t\ell}\right|^s \leq C a^{\ell},$$

for a suitable constant $C > 0$. The dependence coefficient $\theta_{\ell,s}$ defined in Liu and Wu (2010), Section 2, can be bounded by

$$\theta_{\ell,s}^s \leq C\left[a^{\ell} + a^{\ell+1}\right].$$

Then Assumption $(C2)'$ follow. Applying Theorem 2.2 in Liu and Wu (2010), we deduce that the convergence given in Corollary 2 holds but with Δ_n replaced by

$$\overline{\Delta}_n = \sqrt{nb} \sup_{v \in \mathcal{I}} \frac{\left|f_X^{\#}(v) - \mathbb{E} f_X^{\#}(v)\right|}{\sqrt{f_X(v) \int K^2(u) du}}.$$

But since f_X has a bounded second derivative and $nb^4 \to 0$, we have $\sup_{v \in I}\left|\mathbb{E} f_X^{\#}(v) - f_X(v)\right| = o\left(n^{-1/8}\right)$. Moreover, from Theorem 1 (2.), we have $\sqrt{nb}\sup_{v \in I}\left|\hat{f}_X(v) - f_X(v)\right| = O_{\mathbb{P}}\left(n^{-1/2}\right)$. We then deduce that

$$\overline{\Delta}_n - \Delta_n = O_{\mathbb{P}}\left(\sqrt{b} + n^{-1/8}\right).$$

Hence the same limit in distribution also holds for Δ_n which proves the result.□

11.2 Simulation study

We investigate the performances of our test for a specific example. For homoscedastic data, our test is very similar to that of Kim et al. (2015) and we prefer to give a numerical comparison for ARCH processes. In this section, we consider the problem of testing H_0: X follows an ARCH(3). Under H_0, the dynamic of the process is given by

$$X_t = 0.1 + \varepsilon_t \sqrt{0.1 + 0.3\left(X_{t-1} - 0.1\right)^2 + 0.3\left(X_{t-2} - 0.1\right)^2 + 0.3\left(X_{t-3} - 0.1\right)^2},$$

and ε_t follows a standard Gaussian distribution. For this example, we have shown that the assumptions **A1-A8** are satisfied for any compact interval I which does not contain 0.1. We choose $\mathcal{I} = [-1, 0] \cup [0.2, 1]$. Our goal here is to see if our test can detect some departures from the null hypothesis in particular power ARCH dynamics:

$$X_t = 0.1 + \varepsilon_t\left(0.1 + 0.3\left|X_{t-1} - 0.1\right|^{\delta} + 0.3\left|X_{t-2} - 0.1\right|^{\delta} + 0.3\left|X_{t-3} - 0.1\right|^{\delta}\right)^{1/\delta}.$$

Under H_1, we assume that $\delta = 1$. In the latter case, the volatility is linear w.r.t. absolute past returns. $\delta = 2$ corresponds to the standard ARCH process. Deciding which power of the log returns has to be included in the volatility has been one of the important question in finance and led Ding et al. (1993) to consider power ARCH models as alternative to the standard ARCH processes. As pointed out in Kim et al. (2015), the type of convergence given in our Corollary 2 is quite slow and for their statistics, the authors fix the critical values for the test by simulating many times a pivotal statistics under a particular scenario for which the asymptotic behavior of the same statistics is still valid. In our conditionally heteroscedastic case, we found that the critical values obtained by this Monte-Carlo method can be quite sensitive to the simulation setup and we prefer to use the block bootstrap method discussed in Kunsch (1989). Suppose that a bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ is available. We then replace the statistics Δ_n by

$$\Delta_n^* = \sqrt{nb} \sup_{v \in \mathcal{I}} \frac{\left| f_X^{\#,*}(v) - \hat{f}_X^*(v) - f_X^{\#}(v) + \hat{f}_X(v) \right|}{\sqrt{\hat{f}_X^*(v) \int K^2(u) du}},$$

where $f_X^{\#,*}$ and $\hat{f}_X^*$ are the versions of $f_X^{\#}$ and $\hat{f}_X$ computed with the bootstrap sample. In the i.i.d. case and when the density is known under H_0, the asymptotic validity of such bootstrap is discussed in Bickel and Ren (2001) (see Example 7). We estimate the parameters (location and ARCH parameters) using weighted least squares. The block bootstrap method requires to fix a size ℓ for the block which typically satisfies $\ell \to \infty$ and $\ell/n \to 0$. For estimating standard errors, the optimal rate for ℓ is $n^{1/3}$. See for instance Bühlmann and Künsch (1999). In our simulation example, we fix $\ell = 15$ and the number of bootstrap samples is $B = 1000$ and we use 500 replications to approximate the true probabilities. Smaller or larger blocs did not give much better results here. Tables 1 and 2 report the coverage probabilities when $b = cn^{-0.3}$. One can see the accuracy of the results depend of the bandwidth. The optimal value for the constant seems to be $c = 1.25$. Under the alternative, we compare the power of our test with that of Kim et al. (2015) which consists in estimating the density of the data normalized by a preliminary estimation of the volatility. We fix $\alpha = 10\%$ and the critical values for the competitive test are obtained by simulating their statistics 5000 times under the null hypothesis. Then the level of the second test is exactly fixed. Results are reported in Table 3. One can see that the results are quite sensitive to the bandwidth. For our best bandwidth for the coverage probabilities (the constant is $c = 1.25$), we see that the power of our test is much larger than the optimal power given by the test of Kim et al. (2015). Other bandwidth parameters did not improve the results. In this conditionally heteroscedastic example, one can see that working with the original data can be more interesting than first normalizing the data. However, bandwidth selection seems to be crucial to obtain satisfying results. Note that bandwidth selection has not been discussed in Kim et al. (2015). Finding results in this direction is beyond the scope of this paper.

Table 1: Coverage probabilities for the bootstrap approximation ($n = 300$)

	$c = 0.5$	$c = 0.75$	$c = 1$	$c = 1.25$	$c = 1.5$
$1 - \alpha = 0.90$	97.4	97.2	95.6	91.2	81.4
$1 - \alpha = 0.95$	98.6	99.2	99.2	97	91.6

Table 2: Coverage probabilities (in %) for the bootstrap approximation ($n = 600$)

	$c = 0.5$	$c = 0.75$	$c = 1$	$c = 1.25$	$c = 1.5$
$1 - \alpha = 0.90$	97	95	93.4	90.8	81.8
$1 - \alpha = 0.95$	99	98	98.2	95.8	91.6

Table 3: Power of the test (in %) for $\delta = 1$, $n = 600$ and $\alpha = 0.1$

	$c = 0.5$	$c = 0.75$	$c = 1$	$c = 1.25$	$c = 1.5$
Test based on Δ_n	6.2	8.6	28.2	70.2	98.4
Test based on Ξ_n	20.4	19	10.4	10	10.6

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